ME1401 – INTRODUCTION OF FINITE ELEMENT ANALYSIS

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ME 1401 INTRODUCTION OF FINITE ELEMENT ANALYSIS

Unit – I Fundamental Concepts

Syllabus

Historical background – Matrix approach – Application to the continuum – Discretisation – Matrix algebra – Gaussian elimination – Governing equations for continuum – Classical Techniques in FEM – Weighted residual method – Ritz method

- **General Methods of the Finite Element Analysis**

  1. Force Method – Internal forces are considered as the unknowns of the problem.

  2. Displacement or stiffness method – Displacements of the nodes are considered as the unknowns of the problem.

- **General Steps of the Finite Element Analysis**

  Discretization of structure > Numbering of Nodes and Elements > Selection of Displacement function or interpolation function > Define the material behavior by using Strain – Displacement and Stress – Strain relationships > Derivation of element stiffness matrix and equations > Assemble the element equations to obtain the global or total equations > Applying boundary conditions > Solution for the unknown displacements > computation of the element strains and stresses from the nodal displacements > Interpret the results (post processing).

- **Boundary Conditions**

  It can be either on displacements or on stresses. The boundary conditions on displacements to prevail at certain points on the boundary of the body, whereas the boundary conditions on stresses require that the stresses induced must be in equilibrium with the external forces applied at certain points on the boundary of the body.

- **Consideration During Discretization process**

  Types of element > Size of element > Location of node > Number of elements.

- **Rayleigh – Ritz Method (Variational Approach)**

It is useful for solving complex structural problems. This method is possible only if a suitable functional is available. Otherwise, Galerkin's method of weighted residual is used.

- **Problems (I set)**

1. A simply supported beam subjected to uniformly distributed load over entire span. Determine the bending moment and deflection at midspan by using Rayleigh – Ritz method and compare with exact solutions.

2. A bar of uniform cross section is clamed at one end and left free at another end and it is subjected to a uniform axial load P. Calculate the displacement and stress in a bar by using two terms polynomial and three terms polynomial. Compare with exact solutions.

- **Weighted Residual method**

It is a powerful approximate procedure applicable to several problems. For non – structural problems, the method of weighted residuals becomes very useful. It has many types. The popular four methods are,

1. Point collocation method,

Residuals are set to zero at n different locations X_i, and the weighting function w_i is denoted as $\delta(x - x_i)$.

$$\int \delta(x - xi)\, R\,(x; a_1, a_2, a_3 \ldots an)\, dx = 0$$

2. Subdomain collocation method,

$$w_1 = \begin{cases} 1\, for x in D1 \\ 0\, for x not in D1 \end{cases}$$

3. Least square method,

$$\int [R\,(x; a_1, a_2, a_3 \ldots an)]^2\, dx = \text{minimum.}$$

4. Galerkin's method.

$$w_i = N_i\,(x)$$

$$\int N_i\,(x)\,[R\,(x; a_1, a_2, a_3 \ldots an)]^2\, dx = 0, \qquad i = 1, 2, 3, \ldots n.$$

- **Problems (II set)**

1. The following differential equation is available for a physical phenomenon.

$\frac{d^2y}{dx^2}$ + 50 = 0, $0 \le x \le 10$. Trial function is $y = a_1 x(10-x)$. Boundary conditions are y (0) = 0 and y (10) = 0. Find the value of the parameter a_1 by the following methods, (1) Point collocation method, (2) Subdomain collocation method, (3) Least square method and (4) Galerkin's method.

2. The differential equation of a physical phenomenon is given by

$\frac{d^2y}{dx^2} - 10x^2 = 5$. Obtain two term Galerkin solution b using the trial functions:
$N_1(x) = x(x-1)$; $N_2(x) = x^2(x-1)$; $0 \le x \le 1$. Boundary conditions are y (0) = 0 and y (1) = 0.

- **Matrix Algebra**

Equal matrix: Two matrixes are having same order and corresponding elements are equal.

Diagonal matrix: Square matrix in which all the elements other than the diagonal are zero.

Scalar matrix: Square matrix in which all the elements are equal.

Unit matrix: All diagonal elements are unity and other elements are zero.

- **Matrix Operation**

Scalar multiplication, Addition and Subtraction of matrices, Multiplication of matrices, Transpose of a matrix, Determinant of a matrix, inverse of a matrix, Cofactor or Adjoint method to determine the inverse of a matrix, Row reduction method (Gauss Jordan method) to determine the inverse of a matrix, Matrix differentiation and matrix integration.

- **Gaussian Elimination Method**

It is most commonly used for solving simultaneous linear equations. It is easily adapted to the computer for solving such equations.

- **Problems (III set)**

1. $3x+y-z = 3$, $2x-8y+z = -5$, $x-2y+9z = 8$. Solve by using Gauss – Elimination method.

2. $2a+4b+2c = 15$, $2a+b+2c = -5$, $4a+b-2c = 0$. Solve the equations by using Gauss – Elimination method.

- **Advantages of Finite Element Method**

1. FEM can handle irregular geometry in a convenient manner.
2. Handles general load conditions without difficulty
3. Non – homogeneous materials can be handled easily.
4. Higher order elements may be implemented.

- **Disadvantages of Finite Element Method**

1. It requires a digital computer and fairly extensive
2. It requires longer execution time compared with FEM.
3. Output result will vary considerably.

- **Applications of Finite Element Analysis**

Structural Problems:

1. Stress analysis including truss and frame analysis
2. Stress concentration problems typically associated with holes, fillets or other changes in geometry in a body.
3. Buckling Analysis: Example: Connecting rod subjected to axial compression.
4. Vibration Analysis: Example: A beam subjected to different types of loading.

Non - Structural Problems:

1. Heat Transfer analysis:

   Example: Steady state thermal analysis on composite cylinder.
2. Fluid flow analysis:

   Example: Fluid flow through pipes.

Unit – II One Dimension Problems

Syllabus

Finite element modeling – Coordinates and shape functions- Potential energy approach – Galarkin approach – Assembly of stiffness matrix and load vector – Finite element equations – Quadratic shape functions – Applications to plane trusses

- **One Dimensional elements**

  Bar and beam elements are considered as One Dimensional elements. These elements are often used to model trusses and frame structures.

- **Bar, Beam and Truss**

  Bar is a member which resists only axial loads. A beam can resist axial, lateral and twisting loads. A truss is an assemblage of bars with pin joints and a frame is an assemblage of beam elements.

- **Stress, Strain and Displacement**

  Stress is denoted in the form of vector by the variable x as σ_x, Strain is denoted in the form of vector by the variable x as e_x, Displacement is denoted in the form of vector by the variable x as u_x.

- **Types of Loading**

  (1) Body force (f)

  It is a distributed force acting on every elemental volume of the body. Unit is Force / Unit volume. Ex: Self weight due to gravity.

  (2) Traction (T)

  It is a distributed force acting on the surface of the body. Unit is Force / Unit area. But for one dimensional problem, unit is Force / Unit length. Ex: Frictional resistance, viscous drag and Surface shear.

  (3) Point load (P)

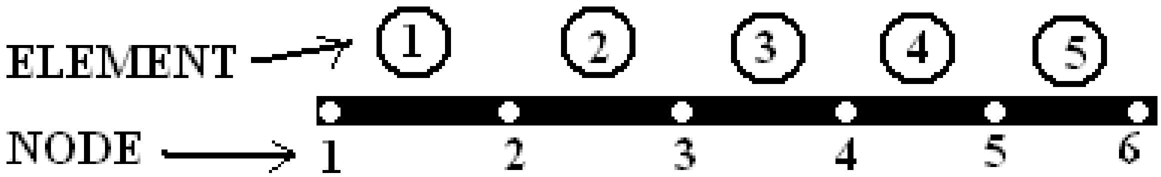
  It is a force acting at a particular point which causes displacement.

- **Finite Element Modeling**

It has two processes.

(1) Discretization of structure

(2) Numbering of nodes.



- **CO – ORDINATES**

(A) Global co – ordinates, (B) Local co – ordinates and (C) Natural co – ordinates.

- **Natural Co – Ordinate (ε)**

$$\varepsilon = \frac{pc}{\left(\frac{x2 - x1}{2}\right)}$$

Integration of polynomial terms in natural co – ordinates for two dimensional elements can be performed by using the formula,

$$\oint (L_1)^{\alpha} (L_2)^{\beta} (L_3)^{\gamma}\, dA = \frac{\alpha! \beta! \gamma!}{(\alpha + \beta + \gamma)!} X 2A$$

- **Shape function**

$N_1N_2N_3$ are usually denoted as shape function. In one dimensional problem, the displacement

$u = \Sigma N_i u_i = N_1 u_1$

For two noded bar element, the displacement at any point within the element is given by,

$u = \Sigma N_i u_i = N_1 u_1 + N_2 u_2$

For three noded triangular element, the displacement at any point within the element is given by,

$$u = \Sigma N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = \Sigma N_i v_i = N_1 v_1 + N_2 v_2 + N_3 v_3$$

Shape function need to satisfy the following

(a) First derivatives should be finite within an element; (b) Displacement should be continuous across the element boundary.

- **Polynomial Shape function**

Polynomials are used as shape function due to the following reasons,

(1) Differentiation and integration of polynomials are quite easy.

(2) It is easy to formulate and computerize the finite element equations.

(3) The accuracy of the results can be improved by increasing the order of the polynomial.

- **Stiffness Matrix [K]**

Stiffness Matrix [K] = $\int_V [B]^T [D][B] dv$

- **Properties of Stiffness Matrix**

1. It is a symmetric matrix, 2. The sum of elements in any column must be equal to zero, 3. It is an unstable element. So the determinant is equal to zero.

- **Equation of Stiffness Matrix for One dimensional bar element**

$$[K] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- **Finite Element Equation for One dimensional bar element**

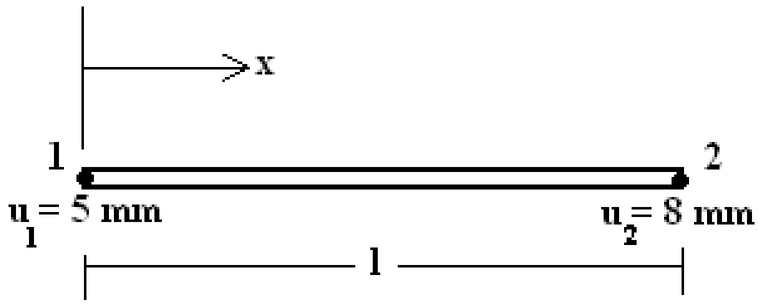
$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

- **The Load (or) Force Vector {F}**

$$\{F\}_e = \frac{\ell A l}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$

- **Problem (I set)**

1. A two noded truss element is shown in figure. The nodal displacements are u_1 = 5 mm and u_2 = 8 mm. Calculate the displacement at x = ¼, 1/3 and ½.



- **Trusses**

It is made up of several bars, riveted or welded together. The following assumptions are made while finding the forces in a truss,

(a) All members are pin joints, (b) The truss is loaded only at the joints, (c) The self – weight of the members is neglected unless stated.

- **Stiffness Matrix [K] for a truss element**

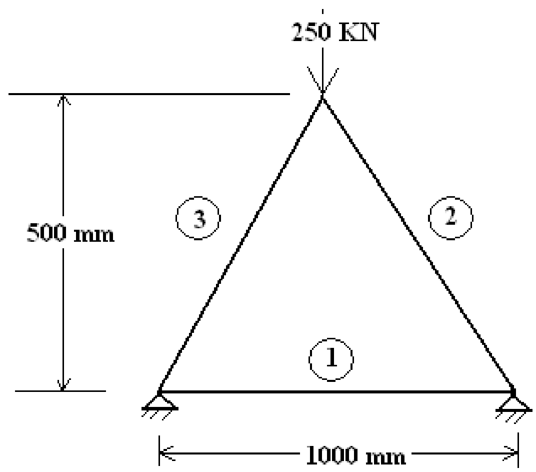
$$[K] = \frac{A_e E_e}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

- **Finite Element Equation for Two noded Truss element**

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \frac{A_e E_e}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

- **Problem (II set)**

1. Consider a three bar truss as shown in figure. It is given that E = 2 x 10^5 N/mm^2. Calculate (a) Nodal displacement, (b) Stress in each member and (c) Reactions at the support. Take Area of element 1 = 2000 mm^2, Area of element 2 = 2500 mm^2, Area of element 3 = 2500 mm^2.



- **The Galerkin Approach**

Stiffness Matrix $[K] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

- **Types of beam**

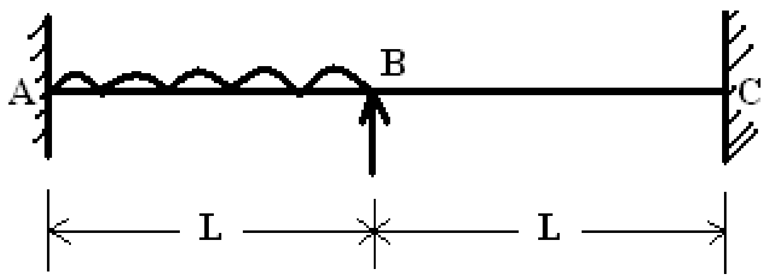
1. Cantilever beam, 2. Simply Supported beam, 3. Over hanging beam, 4. Fixed beam and 5. Continuous beam.

- **Types of Transverse Load**

1. Point or Concentrated Load, 2. Uniformly Distributed Load and 3. Uniformly Varying Load.

- **Problem (III set)**

1. A fixed beam of length 2L m carries a uniformly distributed load of w (N/m) which runs over a length of L m from the fixed end. Calculate the rotation at Point B.



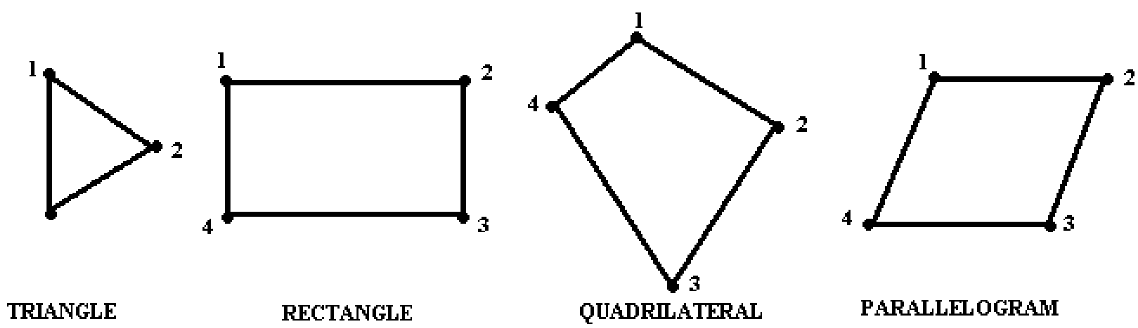
Unit – III Two Dimension Problems – Scalar variable Problems

Syllabus

Finite element modeling – CST & LST elements – Elements equations – Load vectors and boundary conditions – Assembly – Applications to scalar variable problems such as torsion, heat transfer.

- **Two dimensional elements**

Two dimensional elements are defined by three or more nodes in a two dimensional plane (i.e., x, y plane). The basic element useful for two dimensional analysis is the triangular element.



- **Plane Stress and Plane Strain**

The 2d element is extremely important for the Plane Stress analysis and Plane Strain analysis.

Plane Stress Analysis:

It is defined to be a state of stress in which the normal stress (σ) and shear stress (τ) directed perpendicular to the plane are assumed to be zero.

Plane Strain Analysis:

It is defined to be a state of strain in which the normal to the xy plane and the shear strain are assumed to be zero.

- **Finite Element Modeling**

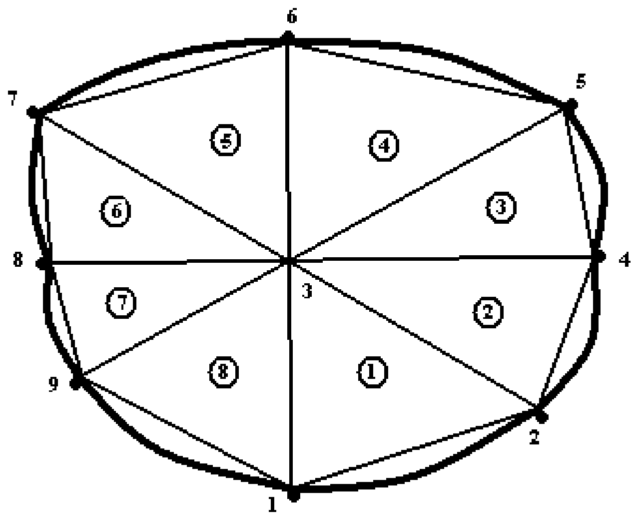
It consists of 1. Discretization of structure and 2. Numbering of nodes.

1. Discretization:

The art of subdividing a structure into a convenient number of smaller components is known as discretization.

2. Numbering of nodes:

In one dimensional problem, each node is allowed to move only in ± x direction. But in two dimensional problem, each node is permitted to move in the two directions i.e., x and y.

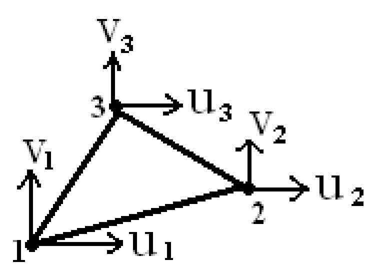


The element connectivity table for the above domain is explained as table.

Element (e)	Nodes
(1)	123
(2)	234
(3)	435
(4)	536
(5)	637
(6)	738
(7)	839
(8)	931

➢ **Constant Strain Triangular (CST) Element**

A three noded triangular element is known as constant strain triangular (CST) element. It has six unknown displacement degrees of freedom (u_1v_1, u_2v_2, u_3v_3).



- **Shape function for the CST element**

  Shape function $N_1 = (p_1 + q_1x + r_1y) / 2A$

  Shape function $N_2 = (p_2 + q_2x + r_2y) / 2A$

  Shape function $N_3 = (p_3 + q_3x + r_3y) / 2A$

- **Displacement function for the CST element**

  Displacement function u = $\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N1 & 0 & N2 & 0 & N3 & 0 \\ 0 & N1 & 0 & N2 & 0 & N3 \end{bmatrix} X \begin{Bmatrix} u1 \\ v1 \\ u2 \\ v2 \\ u3 \\ v3 \end{Bmatrix}$

- **Strain – Displacement matrix [B] for CST element**

  Strain – Displacement matrix [B] = $\frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$

  Where, $q_1 = y_2 - y_3$ $\quad$ $r_1 = x_3 - x_2$

  $q_2 = y_3 - y_1$ $\quad$ $r_2 = x_1 - x_3$

  $q_3 = y_1 - y_2$ $\quad$ $r_3 = x_2 - x_1$

- **Stress – Strain relationship matrix (or) Constitutive matrix [D] for two dimensional element**

$$[D] = \frac{E}{(1+v)\quad(1-2v)}\begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

- **Stress – Strain relationship matrix for two dimensional plane stress problems**

  The normal stress σ_z and shear stresses τ_{xz}, τ_{yz} are zero.

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

- **Stress – Strain relationship matrix for two dimensional plane strain problems**

  Normal strain e_z and shear strains e_{xz}, e_{yz} are zero.

$$[D] = \frac{E}{(1+v)\ (1-2v)}\begin{bmatrix} (1-v) & v & 0 \\ v & (1-v) & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

- **Stiffness matrix equation for two dimensional element (CST element)**

  Stiffness matrix $[k] = [B]^T [D] [B]\, A\, t$

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

For plane stress problems,

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

For plane strain problems,

$$[D] = \frac{E}{(1+v)\ (1-2v)} \begin{bmatrix} (1-v) & v & 0 \\ v & (1-v) & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

- **Temperature Effects**

Distribution of the change in temperature (ΔT) is known as strain. Due to the change in temperature can be considered as an initial strain e_0.

$$\sigma = D\ (Bu - e_0)$$

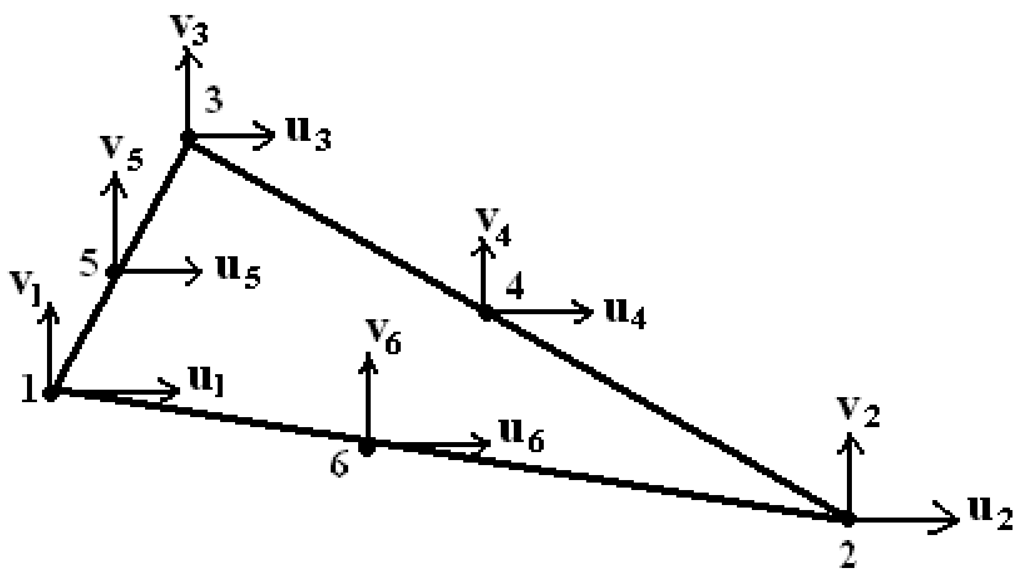
- **Galerkin Approach**

$$\text{Stiffness matrix } [K]_e = [B]^T\ [D][B]\ A\ t.$$

$$\text{Force Vector } \{F\}_e = [K]_e\ \{u\}$$

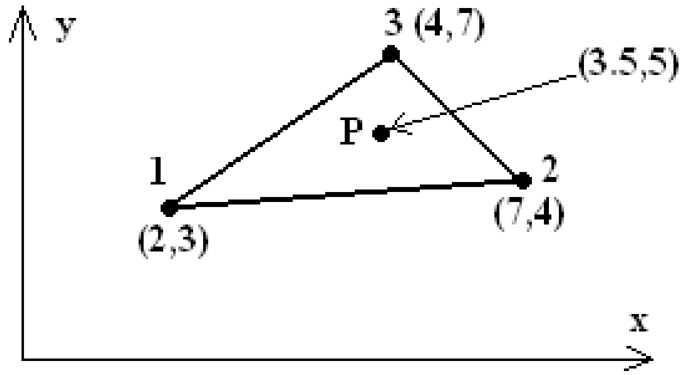
- **Linear Strain Triangular (LST) element**

A six noded triangular element is known as Linear Strain Triangular (LST) element. It has twelve unknown displacement degrees of freedom. The displacement functions of the element are quadratic instead of linear as in the CST.



- **Problem (I set)**

1. Determine the shape functions N_1, N_2 and N_3 at the interior point P for the triangular element for the given figure.

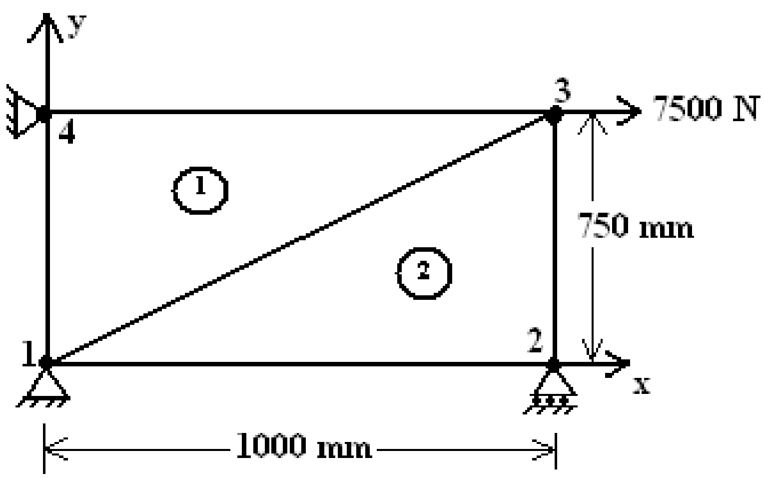


The two dimensional propped beam shown in figure. It is divided into two CST elements. Determine the nodal displacement and element stresses using plane stress conditions. Body force is neglected in comparison with the external forces.

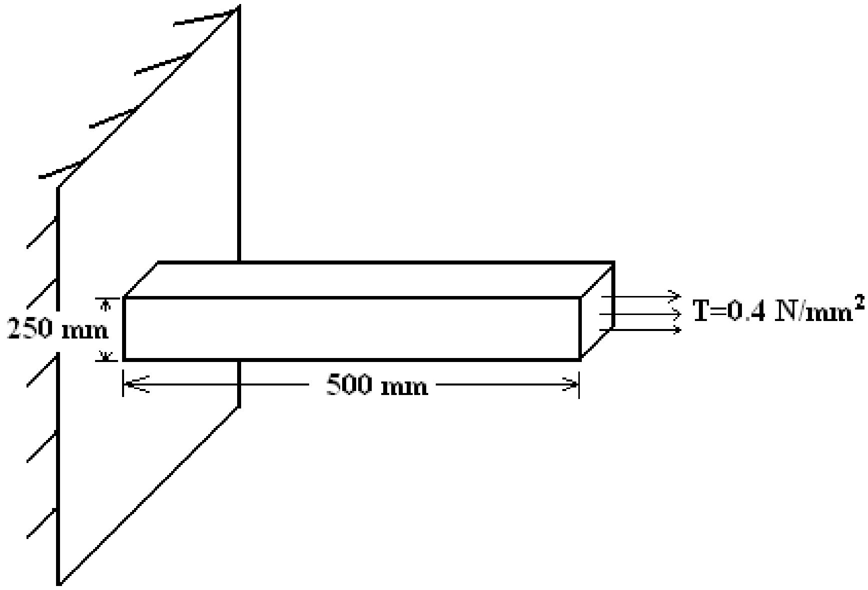
Take, Thickness (t) = 10mm,

Young's modulus (E) = $2x10^5$ N/mm^2,

Poisson's ratio (v) = 0.25.



3. A thin plate is subjected to surface traction as in figure. Calculate the global stiffness matrix.



- **Scalar variable problems**

In structural problems, displacement at each nodal point is obtained. By using these displacement solutions, stresses and strains are calculated for each element. In structural problems, the unknowns (displacements) are represented by the components of vector field. For example, in a two dimensional plate, the unknown quantity is the vector field u(x, y), where u is a (2x1) displacement vector.

- **Equation of Temperature function (T) for one dimensional heat conduction**

$$\text{Temperature (T)} = N_1T_1 + N_2T_2$$

- **Equation of Shape functions (N_1 & N_2) for one dimensional heat conduction**

$$\mathbf{N}_1 = \frac{l - x}{l}$$

$$\mathbf{N}_2 = \frac{x}{l}$$

- **Equation of Stiffness Matrix (K) for one dimensional heat conduction**

$$[K_c] = \frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- **Finite Element Equations for one dimensional heat conduction**

Case (i): One dimensional heat conduction with free end convection

$$\left[\frac{Ak}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right]\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = hT_\infty A\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

Case (ii): One dimensional element with conduction, convection and internal heat generation.

$$\left[\frac{Ak}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{hPl}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right]\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \frac{QAl + PhT_\infty l}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

- **Finite element Equation for Torsional Bar element**

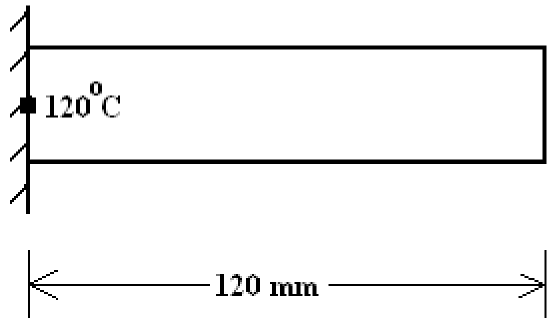
$$\begin{Bmatrix}M_{1x}\\ M_{2x}\end{Bmatrix} = \frac{GJ}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\phi_{1x}\\ \phi_{2x}\end{Bmatrix}$$

Where, Stiffness matrix [K] = $\frac{GJ}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

- **Problem (II set)**

1. An Aluminium alloy fin of 7 mm thick and 50 mm long protrudes from a wall, which is maintained at 120°C. The ambient air temperature is 22°C. The heat transfer coefficient and thermal conductivity of fin material are 140 W/m^{2}K and 55 W/mK respectively. Determine the temperature distribution of fin.

2. Calculate the temperature distribution in a one dimension fin with physical properties given in figure. The fin is rectangular in shape and is 120 mm long, 40mm wide and 10mm thick. Assume that convection heat loss occurs from the end of the fin. Use two elements. Take k = 0.3W/mm°C, h = 1 x 10-3 W/ mm^2°C, T=20°C.


120°C
120 mm

Unit – IV AXISYMMETRIC CONTINUUM

Syllabus

Axisymmetric formulation – Element stiffness matrix and force vector – Galarkin approach – Body forces and temperature effects – Stress calculations – Boundary conditions – Applications to cylinders under internal or external pressures – Rotating discs

- **Elasticity Equations**

  Elasticity equations are used for solving structural mechanics problems. These equations must be satisfied if an exact solution to a structural mechanics problem is to be obtained. The types of elasticity equations are

  1. Strian – Displacement relationship equations

$$e_x = \frac{\partial u}{\partial x};\ e_y = \frac{\partial v}{\partial y};\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x};\ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x};$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.$$

e_x – Strain in X direction, e_y – Strain in Y direction.

γ_{xy} - Shear Strain in XY plane, γ_{xz} - Shear Strain in XZ plane,

γ_{yz} - Shear Strain in YZ plane

  2. Sterss – Strain relationship equation

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_x \\ e_x \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

σ – Stress, τ – Shear Stress, E – Young's Modulus, v – Poisson's Ratio,

e – Strain, γ - Shear Strain.

3. Equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0 ; \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} + B_y = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + B_z = 0$$

σ – Stress, τ – Shear Stress, B_x - Body force at X direction,

B_y - Body force at Y direction, B_z - Body force at Z direction.

4. Compatibility equations

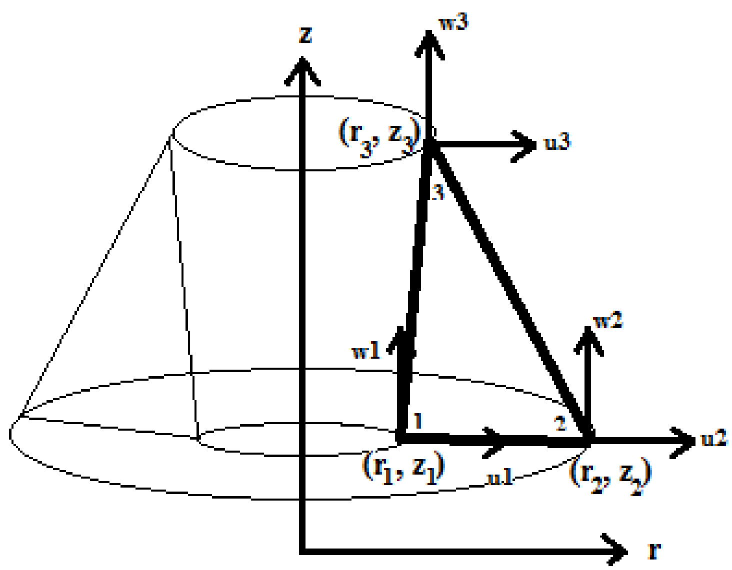
There are six independent compatibility equations, one of which is

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} .$$

The other five equations are similarly second order relations.

- **Axisymmetric Elements**

Most of the three dimensional problems are symmetry about an axis of rotation. Those types of problems are solved by a special two dimensional element called as axisymmetric element.



- **Axisymmetric Formulation**

The displacement vector u is given by

$$u(r,z) = \begin{Bmatrix} u \\ w \end{Bmatrix}$$

The stress σ is given by

$$Stress, \{\sigma\} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix}$$

The strain e is given by

$$Strain, \{e\} = \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix}$$

- **Equation of shape function for Axisymmetric element**

Shape function,

$$N_1 = \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A};\ N_2 = \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A};\ N_3 = \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A}$$

$\alpha_1 = r_2z_3 - r_3z_2$; $\alpha_2 = r_3z_1 - r_1z_3$; $\alpha_3 = r_1z_2 - r_2z_1$

$\beta_1 = z_2 - z_3$; $\beta_2 = z_3 - z_1$; $\beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - r_2$; $\gamma_2 = r_1 - r_3$; $\gamma_3 = r_2 - r_1$

$2A = (r_2z_3 - r_3z_2) - r_1(r_3z_1 - r_1z_3) + z_1(r_1z_2 - r_2z_1)$

- **Equation of Strain – Displacement Matrix [B] for Axisymmetric element**

$$[B]=\frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

$$r=\frac{r1+r2+r3}{3}$$

- **Equation of Stress – Strain Matrix [D] for Axisymmetric element**

$$[D]=\frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

- **Equation of Stiffness Matrix [K] for Axisymmetric element**

$$[K]=2\prod rA[B]^T[D][B]$$

$r=\frac{r1+r2+r3}{3}$; A = (½) bxh

- **Temperature Effects**

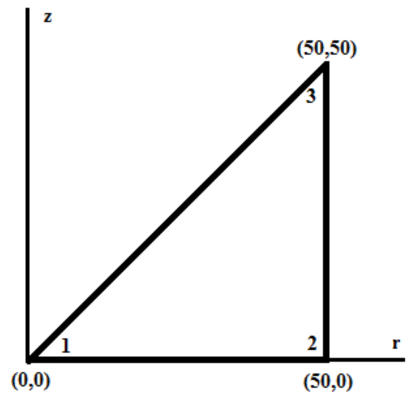
The thermal force vector is given by

$$\{f\}_t=2\prod rA[B][D]\{e\}_t$$

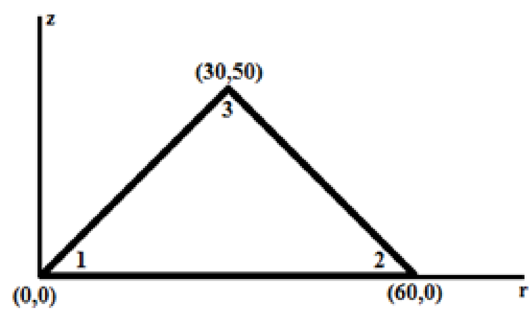
$$\{f\}_t = \begin{Bmatrix} F_1 u \\ F_1 w \\ F_2 u \\ F_2 w \\ F_3 u \\ F_3 w \end{Bmatrix}$$

- **Problem (I set)**

1. For the given element, determine the stiffness matrix. Take E=200GPa and υ= 0.25.



2. For the figure, determine the element stresses. Take E=2.1x10^{5}N/mm^2 and υ= 0.25. The co – ordinates are in mm. The nodal displacements are u_1=0.05mm, w_1=0.03mm, u_2=0.02mm, w_2=0.02mm, u_3=0.0mm, w_3=0.0mm.



3. A long hollow cylinder of inside diameter 100mm and outside diameter 140mm is subjected to an internal pressure of 4N/mm2. By using two elements on the 15mm length, calculate the displacements at the inner radius.

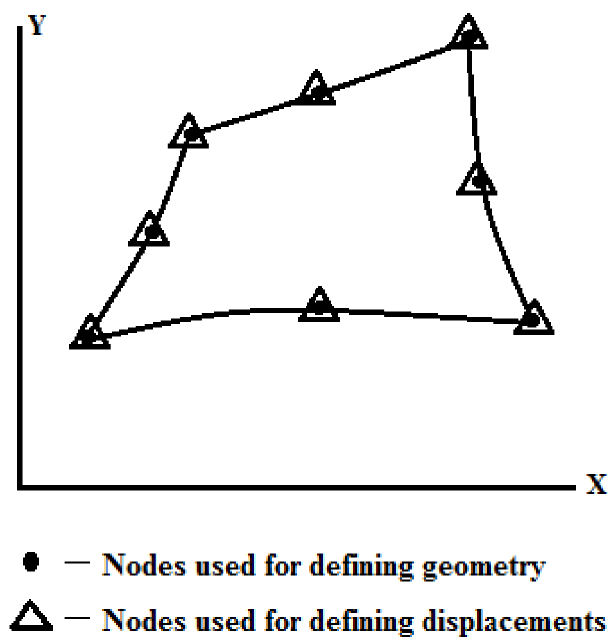
UNIT – V ISOPARAMETRIC ELEMENTS FOR TWO DIMENSIONAL CONTINUUM

Syllabus

The four node quadrilateral – Shape functions – Element stiffness matrix and force vector – Numerical integration - Stiffness integration – Stress calculations – Four node quadrilateral for axisymmetric problems.

- **Isoparametric element**

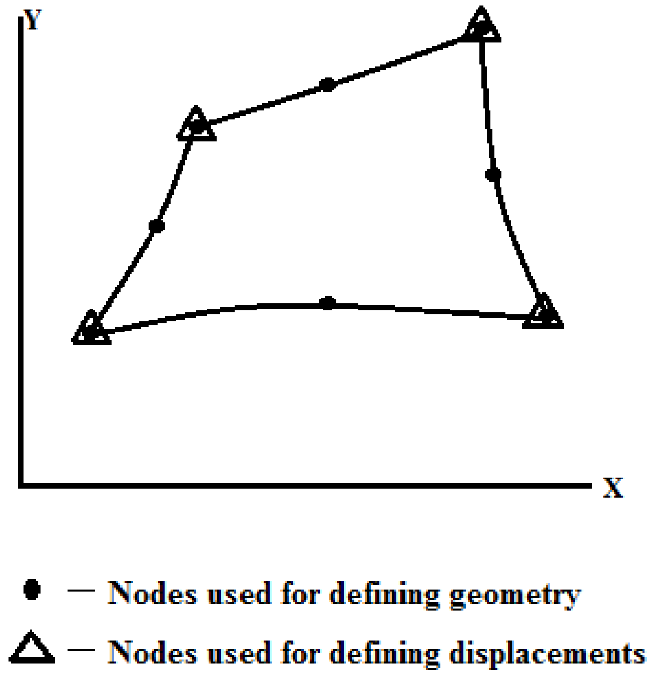
  Generally it is very difficult to represent the curved boundaries by straight edge elements. A large number of elements may be used to obtain reasonable resemblance between original body and the assemblage. In order to overcome this drawback, isoparametric elements are used.



If the number of nodes used for defining the geometry is same as number of nodes used defining the displacements, then it is known as isoparametric element.

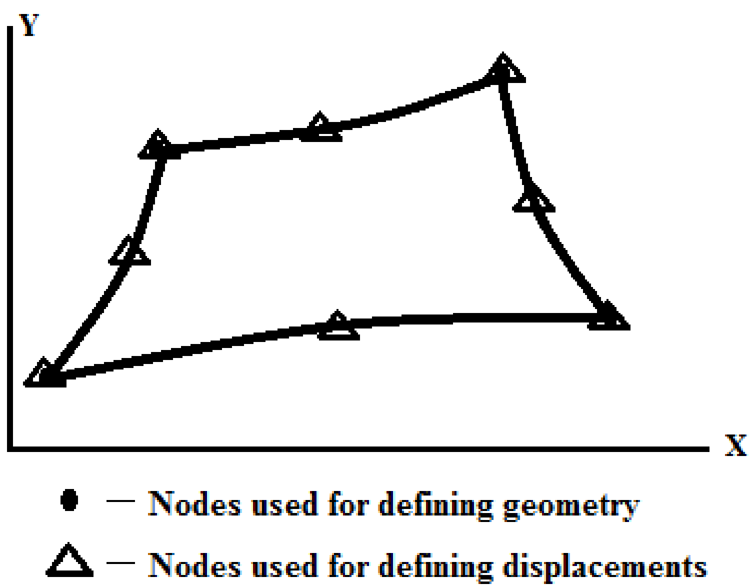
- **Superparametric element**

  If the number of nodes used for defining the geometry is more than number of nodes used for defining the displacements, then it is known as superparametric element.



- **Subparametric element**

  If the number of nodes used for defining the geometry is less than number of nodes used for defining the displacements, then it is known as subparametric element.



- **Equation of Shape function for 4 noded rectangular parent element**

$$u=\begin{Bmatrix} x \\ y \end{Bmatrix}=\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}\begin{Bmatrix} x_1 \\ y_2 \\ x_1 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

N_1=1/4(1-Ɛ) (1-ƞ); N_2=1/4(1+Ɛ) (1-ƞ); N_3=1/4(1+Ɛ) (1+ƞ); N_4=1/4(1-Ɛ) (1+ƞ).

- **Equation of Stiffness Matrix for 4 noded isoparametric quadrilateral element**

$$[K]=t\int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]|J|\partial\varepsilon\partial\eta$$

$$[J]=\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix};$$

$$J_{11}=\frac{1}{4}[-(1-\eta)x_1+(1-\eta)x_2+(1+\eta)x_3-(1+\eta)x_4];$$

$$J_{12}=\frac{1}{4}[-(1-\eta)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4];$$

$$J_{21}=\frac{1}{4}[-(1-\varepsilon)x_1-(1+\varepsilon)x_2+(1+\varepsilon)x_3+(1-\varepsilon)x_4];$$

$$J_{22}=\frac{1}{4}[-(1-\varepsilon)y_1-(1+\varepsilon)y_2+(1+\varepsilon)y_3+(1-\varepsilon)y_4];$$

$$[B]=\frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}\begin{bmatrix} \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & 0 & \frac{\partial N_3}{\partial \varepsilon} & 0 & \frac{\partial N_4}{\partial \varepsilon} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & 0 & \frac{\partial N_3}{\partial \varepsilon} & 0 & \frac{\partial N_4}{\partial \varepsilon} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix}$$

$$[D]=\frac{E}{(1-v^2)}\begin{bmatrix} 1-v & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$, for plane stress conditions;

$$[D]=\frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$, for plane strain conditions.

- **Equation of element force vector**

$$\{F\}_e = [N]^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix};$$

N – Shape function, F_x – load or force along x direction,

F_y – load or force along y direction.

- **Numerical Integration (Gaussian Quadrature)**

The Gauss quadrature is one of the numerical integration methods to calculate the definite integrals. In FEA, this Gauss quadrature method is mostly preferred. In this method the numerical integration is achieved by the following expression,

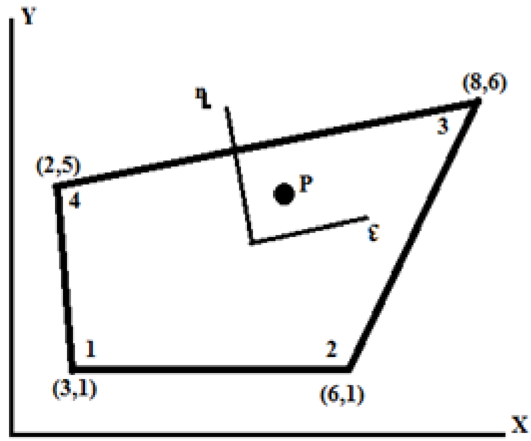
$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

Table gives gauss points for integration from -1 to 1.

Number of Points n	Location x_i	Corresponding Weights w_i
1	$\mathbf{x_1 = 0.000}$	**2.000**
2	$\mathbf{x_1, x_2} = \pm\sqrt{\frac{1}{3}} = \pm 0.577350269189$	**1.000**
3	$\mathbf{x_1, x_3} = \pm\sqrt{\frac{3}{5}} = \pm 0.774596669241$ $\mathbf{x_2 = 0.000}$	$\frac{5}{9} = 0.555555$ $\frac{8}{9} = 0.888888$
4	$\mathbf{x_1, x_4} = \pm 0.8611363116$ $\mathbf{x_2, x_3} = \pm 0.3399810436$	**0.3478548451** **0.6521451549**

- **Problem (I set)**

1. Evaluate $I = \int_{-1}^{1} \cos\frac{\pi x}{2} dx$, by applying 3 point Gaussian quadrature and compare with exact solution.
2. Evaluate $I = \int_{-1}^{1} \left[3e^x + x^2 + \frac{1}{x+2}\right] dx$, using one point and two point Gaussian quadrature. Compare with exact solution.
3. For the isoparametric quadrilateral element shown in figure, determine the local co –ordinates of the point P which has Cartesian co-ordinates (7, 4).



4. A four noded rectangular element is in figure. Determine (i) Jacobian matrix, (ii) Strain – Displacement matrix and (iii) Element Stresses. Take $E=2\mathrm{x}10^5 N/mm^2$, $\upsilon= 0.25$, $u=[0,0,0.003,0.004,0.006, 0.004,0,0]^T$, $\varepsilon= 0$, $\eta=0$. Assume plane stress condition.